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A Family of M-theory Flows with Four Supersymmetries

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Abstract

We use the techniques of "algebraic Killing spinors" to obtain a family of holographic flow solutions with four supersymmetries in M-theory. The family of supersymmetric backgrounds constructed here includes the non-trivial flow to the $(2+1)$-dimensional analog of the Leigh-Strassler fixed point as well as generalizations that involve the $M2$-branes spreading in a radially symmetric fashion on the Coulomb branch of this non-trivial fixed point theory. In spreading out, these $M2$-branes also appear to undergo dielectric polarization into $M5$-branes. Our results naturally extend the earlier applications of the "algebraic Killing spinor" method and also generalize the harmonic Ansatz in that our entire family of new supersymmetric backgrounds is characterized by the solutions of a single, second-order, non-linear PDE. We also show that our solution is a natural hybrid of special holonomy and the "dielectric deformation" of the canonical supersymmetry projector on the $M2$ branes.

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1. Introduction

The problem of finding, and classifying supersymmetric backgrounds with non-trivial RR-fluxes has been around for many years, but it is only relatively recently that it has begun to be addressed systematically. This problem has taken on new significance because of the important role that fluxes play in supersymmetry breaking backgrounds, and particularly in holographic RG flows. The idea of G-structures has provided a very useful classification framework and has led to new families of solutions [1–5], but this approach is, as yet, not computationally powerful enough to reproduce some of the physically important families of holographic RG flow solutions. There is, however, a closely related approach, that of algebraic Killing spinors, which is somewhat more narrowly focussed, but appears to be computationally efficient [6,7]. It is our purpose here to further develop this technique by finding new families of M-theory flows with *four* supersymmetries. Similar results may also be obtained in IIB supergravity [8].

While the ideas presented here apply rather more generally, we will work within M-theory, and define the Killing spinors to be solutions of:

$$\delta\psi_\mu \;\equiv\; \nabla_\mu\, \epsilon \;+\; \tfrac{1}{144}\Big(\Gamma_\mu{}^{\nu\rho\lambda\sigma} \;-\; 8\,\delta_\mu^\nu\,\Gamma^{\rho\lambda\sigma}\Big)\, F_{\nu\rho\lambda\sigma} \;=\; 0\,. \tag{1.1}$$

The essential idea behind G-structures is to classify the special differential forms that arise in supersymmetric flux compactifications. With algebraic Killing spinors, one tries to characterize the spin bundle of the supersymmetries directly (and algebraically) in terms of the metric. The relationship between these approaches is very simple: Given some Killing spinors, $\epsilon^{(i)}$, there are associated differential forms:

$$\Omega^{(ij)}_{\mu_1\mu_2\ldots\mu_k} \;\equiv\; \bar\epsilon^{(i)}\,\Gamma_{\mu_1\mu_2\ldots\mu_k}\,\epsilon^{(j)}\,. \tag{1.2}$$

Conversely, given all these forms one can reconstruct the spinors. The whole point is that the differential forms satisfy systems of first-order differential equations as a consequence of (1.1)[2]. In addition, Fierz identities give detailed information about (partial) contractions of these differential forms, thereby relating them algebraically. Without background fluxes one finds that these forms are harmonic, and that the two-forms often yield some Kähler or hyper-Kähler structure. One is thus led rapidly the study of cohomology of complex manifolds. In the presence of fluxes the systems of differential equations, and the algebraic relations between the forms is more complicated, but the resulting G-structure is the natural generalization of the ideas of using cohomology and complex structures in the absence of fluxes.

With algebraic Killing spinors the idea is solve (1.1) more directly. In Calabi-Yau manifolds, or with G_2 structures one usually finds that the Killing spinor bundles are trivially defined by an algebraic projection, involving the special differential forms, applied to the complete spin bundle. For the simplest intersecting brane solutions these projections are the familiar projections parallel and perpendicular to the branes. The idea is to generalize this to make Ansätze for the projectors that define the Killing spinor bundles. These projections determine the types of differential forms that emerge from (1.2), and so Ansätze for these projectors must be implicitly the same as selecting the type of G-structure. While it would be very interesting to pursue this line of thought, our purpose here is more computationally oriented. The issue with algebraic Killing spinors is to determine the projector Ansatz, and to do that it is valuable to construct explicit, and illustrative examples. In [6,7] this was done for families of flows with eight supersymmetries, while here we are going to obtain and solve natural Ansätze for projectors leading to solutions with four supersymmetries in M-theory. Among the family of solutions that we will generate here is the M-theory flow [9] to a superconformal fixed point (with an AdS_4 background) with four supersymmetries [10–15]. We will see how this work naturally extends the ideas of [6], and elucidates the structure further. We will also see rather explicitly why previous classification schemes have not led to these flows, and we also suggest how to repair this omission.

As in [6], we will look for solutions that correspond to distributions of branes (with additional fluxes) where the brane distribution depends non-trivially, but arbitrarily upon one "radial" variable, v. The solutions therefore depend upon two variables, u and v, where u is essentially a radial coordinate transverse to the brane distribution. Without the additional fluxes, the solution would be elementary, and can easily be written in terms of a harmonic function, $H(u,v)$. The presence of additional fluxes leads to a rather more complicated family of solutions: This family is also determined entirely by a solution, $g(u,v)$, of a second order PDE in u and v, but this PDE is non-linear. Asymptotically one has $g \to 0$ at large (u,v), and the non-linear PDE has a very simple perturbation expansion:

$$g(u,v) \;=\; \sum_{n=1}^{\infty} g_n(u,v)\, \epsilon^n \,, \tag{1.3}$$

where ϵ is a small parameter. As in [6], one then show that g_1 satisfies a linear, homogeneous, second order PDE, while g_n satisfies an equation with the same linear differential operator sourced by combinations of g_k for $k < n$. In this way our solution generalizes the standard harmonic Ansatz.

The first step is to identify projection operators that reduce the dimension of the relevant spinor space. In [6] this required two projectors to reduce 32 supersymmetries to eight, whereas here we have to construct three projection operators, Π_j, $j = 0,1,2$ to reduce the supersymmetries to four. If the solution one seeks is holographically dual to a theory with a Coulomb branch, then there will be a non-trivial space of moduli for brane probes. This moduli space will be realized as either a conformally Kähler (for four supersymmetries) or conformally hyper-Kähler (for eight supersymmetries) section of the metric. On this section of the metric the supersymmetries will satisfy projection conditions $\Pi_j \epsilon = 0$ where the Π_j have the elementary form:

$$\Pi_j \ = \ \frac{1}{2}\left(\mathbb{1} \ + \ \Gamma^{\mathcal{X}_j}\right), \tag{1.4}$$

where $\Gamma^{\mathcal{X}_j}$ denotes a product of gamma-matrices parallel to the moduli space of the branes. For example, in [6] there was one such projector and it implemented the half-flat condition of the spinors on the hyper-Kähler moduli space. For the flows considered here there is a six-dimensional Kähler moduli space and to reduce to one-quarter supersymmetry one must isolate the spinor singlets under the $SU(3)$ factor of the holonomy group. As is familiar with Calabi-Yau 3-folds, this can be achieved by requiring $\Pi_1 \epsilon = \Pi_2 \epsilon = 0$ for two projectors of the form (1.4), where $\mathcal{X}_j$ is chosen based upon the complex structure. Having found Π_1 and Π_2, one conjectures that, in suitably chosen frames, these projectors remain the same for the complete metric, and not just on the conformally Kähler section. The frames that accomplish this extension to the complete metric are canonically determined because the G-structure associated with $\mathcal{N} = 1$ supersymmetry guarantees that the Kähler structure on the moduli space extends at least to an almost complex structure on the whole spatial part of the metric.

The subtlety is the last projector, Π_0, which, based on the experience of [16,6,7] is the deformation of the canonical projector:

$$\Pi_0 \ = \ \frac{1}{2}\left(\mathbb{1} \ + \ \cos\chi(u,v)\ \Gamma^{123} \ + \ \sin\chi(u,v)\ \Gamma^{*}\right), \tag{1.5}$$

where Γ^* denotes a product (or sum of products) of gamma-matrices with $(\Gamma^*)^2 = \mathbb{1}$. If $\chi(u,v) \equiv 0$ then this the standard projector parallel to the $M2$ branes. One can also derive the form of (1.5) by analyzing known solutions [9] from gauged supergravity. There is, however, a more direct way to deduce the form of Γ^*: Based on [6,7,16,17], we know that the form of (1.5) should be interpreted as a the result of the $M2$ branes becoming dielectrically polarized into a distribution of $M5$ branes. Thus Γ^* must be made up of

terms of the form Γ^{123ABC} for some A, B, C. This, combined with the fact that Π_0 must commute with Π_1 and Π_2, and with the symmetries, determines the form of (1.5).

In section 2 we describe more precisely the class of holographic flows that we are seeking, and then we make the complete Ansatz in section 3. The Ansatz is then solved it section 4, where we show that the entire solution is generated by the solution to a single PDE. While the result is relatively simple, it is a rather complicated task in practice to "un-thread" from (1.1) all the independent equations for the Ansatz functions. However, there are some short-cuts that can be made using some of the simplest of the G-structure equations. This is described in section 5, where we focus the role of the almost complex structure. The surprise is that this structure only exists in ten dimensions: It requires one to include the two spatial coordinates of the brane. We believe that this fact goes some way to explain why previous analyses have not led to the families of solution described here: The deformations in (1.5) lead to almost complex structures on the entire spatial part of the metric, and not merely on the "internal" space.

We conclude by drawing out general ideas from the results presented here and in earlier work. We also describe some general open problems to which our methods might find useful application.

2. The holographic flows

The holographic theory on $M2$-branes is an $\mathcal{N} = 8$ supersymmetric theory with eight scalars, eight fermions and sixteen supercharges. The $AdS_4 \times S^7$ background yields the holographic dual of a strongly coupled, superconformal fixed point [18,19]. One can also think of this field theory as arising through a limit of the Kaluza-Klein reduction of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory on a circle. The extra scalars in three-dimensions come from the components of the gauge fields along the circle and a Wilson line parameter around the circle. From this perspective one can frequently link results obtained for Yang-Mills theory to results for the three-dimensional scalar-fermion theory.

A particular example of such a link is the flow to a new superconformal fixed point, with four supersymmetries, obtained by giving a mass to a single chiral multiplet. For $\mathcal{N} = 4$ Yang-Mills theory this flow was analyzed by Leigh and Strassler [20] and its holographic dual was identified in [21–23]. Being strongly coupled, it is very hard to work with the scalar-fermion theory directly, however there is an exactly corresponding holographic flow that was described in [9]. To be specific, the eight scalars and fermions can be paired into four complex superfields, Φ_k, $k = 1, \ldots, 4$. Giving one of them, say,

Φ_4, a mass leads to an $\mathcal{N} = 2$ supersymmetric flow (four supersymmetries) to a new superconformal fixed point. The vevs of remaining scalars, Φ_k, $k = 1, \ldots, 3$, parametrize the Coulomb branch at this fixed point. A brane-probe analysis [24] of the supergravity solution does indeed reveal a three complex-dimensional space of moduli for the brane probes, and that this moduli space has, as one should expect, a natural Kähler structure.

It is this family of flows that we seek to generalize here. The flow of [9] was obtained from gauged supergravity and leads to *a single point* on the Coulomb branch of the fixed point theory. From the field theory and the brane-probe analysis we know that there should be a family of solutions parametrized by a function of six variables describing a general brane distribution on the Coulomb branch at the fixed point. As a first step to finding this general class, we are going to seek flow solutions that correspond to brane distributions with rotational symmetry on the Coulomb branch: That is, we will seek solutions where the brane density depends only upon a radial coordinate on the moduli space.

3. The Ansatz

3.1. Conventions

Our M-theory conventions are those of [25,26]. The metric is "mostly plus," and we take the gamma-matrices to be

$$\begin{aligned} \Gamma_1 &= -i\,\Sigma_2 \otimes \gamma_9\,, \quad \Gamma_2 = \Sigma_1 \otimes \gamma_9\,, \quad \Gamma_3 = \Sigma_3 \otimes \gamma_9\,, \\ \Gamma_{j+3} &= \mathbb{1}_{2\times 2} \otimes \gamma_j\,, \quad j = 1, \ldots, 8\,, \end{aligned} \tag{3.1}$$

where the Σ_a are the Pauli spin matrices, $\mathbb{1}$ is the Identity matrix, and the γ_j are real, symmetric $SO(8)$ gamma matrices. As a result, the Γ_j are all real, with Γ_1 skew-symmetric and Γ_j symmetric for $j > 2$. One also has:

$$\Gamma^{1\cdots\cdots 11} = \mathbb{1}\,, \tag{3.2}$$

where $\mathbb{1}$ will henceforth denote the 32×32 identity matrix. The gravitino variation will be as in (1.1). With these conventions, sign choices and normalizations, the equations of motion are:

$$\begin{aligned} R_{\mu\nu} + R\,g_{\mu\nu} &= \tfrac{1}{3}\,F_{\mu\rho\lambda\sigma}\,F_{\nu}{}^{\rho\lambda\sigma}\,, \\ \nabla_\mu F^{\mu\nu\rho\sigma} &= -\tfrac{1}{576}\,\varepsilon^{\nu\rho\sigma\lambda_1\lambda_2\lambda_3\lambda_4\tau_1\tau_2\tau_3\tau_4}\,F_{\lambda_1\lambda_2\lambda_3\lambda_4}\,F_{\tau_1\tau_2\tau_3\tau_4}\,. \end{aligned} \tag{3.3}$$

3.2. Background fields

We take the metric to have the general form;

$$
\begin{aligned}
ds_{11}^2 &= e^{2A_0}\,(-dx_0^2 + dx_1^2 + dx_2^2) \;+\; e^{2A_1}\,(du^2 \;+\; u^2\,d\phi^2) \;+\; e^{2A_2}\,dv^2 \\
&\quad +\; v^2\,e^{2A_3}\,\big(d\lambda^2 \;+\; \tfrac{1}{4}\,\sin^2\lambda(\sigma_1^2+\sigma_2^2) \;+\; \tfrac{1}{4}\,\sin^2\lambda\,\cos^2\lambda\,\sigma_3^2\big) \\
&\quad +\; v^2\,e^{2A_2}\big(\,e^{A_4}\big(d\psi - \tfrac{1}{2}\,\sin^2\lambda\,\sigma_3\big) \;+\; e^{A_5}\,d\phi\,\big)^2\,,
\end{aligned}
\tag{3.4}
$$

where $A_0,\ldots,A_5$ are, as yet, arbitrary functions of u and v. This Ansatz is a natural generalization of the metric in [9], and the (u,v) coordinates are related to the variables of [9] via:

$$
u \;=\; e^{A(r)}\,\sqrt{\sinh\chi(r)}\,sin\theta\,, \qquad v \;=\; e^{\frac{1}{2}A(r)}\,\rho(r)\,sin\theta\,. \tag{3.5}
$$

We describe how we arrived at this change of variable at the end of section 5. The important aspect of this change of variables is that it results in conformally flat metric in the (u,ϕ) directions, and as we will describe below, it reveals that the complete spatial section of the metric has an almost complex structure.

One should also note that (u,ϕ) parametrize the internal directions *transverse* to the brane moduli space, and that v, ψ, λ and the σ_j sweep out the moduli space. The middle line of (3.4) is the metric on $\mathbb{CP}^2$, with an $SU(3)$ isometry, and as was noted in [9], this part of the metric could be replaced by any Einstein-Kähler manifold. The radial coordinate on the brane moduli space is thus v, and the solutions we seek have brane distributions that depend solely upon v, and by symmetry, the metric and gauge fields must depend only upon u and v.

We use the frames:

$$
\begin{aligned}
e^j &= e^{A_0}\,dx_{j-1}\,, \quad j=1,2,3\,, \qquad e^4 \;=\; e^{A_1}du\,, \qquad e^5 \;=\; e^{A_2}dv\,, \qquad e^6 \;=\; v\,e^{A_3}\,d\lambda\,, \\
e^{6+j} &= \tfrac{1}{2}\,v\,e^{A_3}\,\sin\lambda\,\sigma_j\,, \quad j=1,2\,, \qquad e^9 \;=\; \tfrac{1}{2}\,v\,e^{A_3}\,\sin\lambda\,\cos\lambda\,\sigma_3\,, \\
e^{10} &= v\,e^{A_2}\big(\,e^{A_4}\big(d\psi - \tfrac{1}{2}\,\sin^2\lambda\,\sigma_3\big) \;+\; e^{A_5}\,d\phi\,\big)\,, \qquad e^{11} \;=\; u\,e^{A_1}\,d\phi\,.
\end{aligned}
\tag{3.6}
$$

In [9] it was found that the tensor gauge field had a natural holomorphic structure on the internal space if it is written in terms of the frames. We therefore respect this structure in making the Ansatz:

$$
\begin{aligned}
A^{(3)} \;=\; p_0(u,v)\,e^1\wedge e^2\wedge e^3 \;&+\; \mathrm{Re}\big[e^{i(\phi+3\,\psi)}\big(p_1(u,v)\,(e^4+i\,e^{11}) \\
&+\; p_2(u,v)\,(e^5+i\,e^{10})\big)\wedge(e^6-i\,e^9)\wedge(e^7-i\,e^8)\big]\,.
\end{aligned}
\tag{3.7}
$$

The holomorphic pairing of frames is straightforward. First recall that the positions in the internal space can be interpreted in terms of vevs of the holomorphic scalars, Φ_k. In the flows we seek the branes are spreading in the Φ_1, Φ_2, Φ_3 directions, while Φ_4 is the field that is given a mass. The latter corresponds to the (u, ϕ) directions, and has a manifest complex structure in (3.4). In the remaining directions there is $\mathbb{CP}^2$ with a Kähler structure that leads to the $(e^6 - i\,e^9) \wedge (e^7 - i\,e^8)$ term in (3.7), while the remaining frames form the last pair. In section 5 we will see how the this holomorphic structure is guaranteed by the G-structure, but for the present we note that the Kähler structure on the brane-probe moduli space must be a conformal multiple of:

$$J_{moduli} \;\equiv\; e^6 \wedge e^9 \;+\; e^7 \wedge e^8 \;-\; e^5 \wedge e^{10}\,. \tag{3.8}$$

3.3. Projectors

Having identified the moduli space and found the complex structure on it, one is immediately led to the following projectors:

$$\Pi_1 \;=\; \frac{1}{2}\left(\mathbb{1} \;+\; \Gamma^{6789}\right), \qquad \Pi_2 \;=\; \frac{1}{2}\left(\mathbb{1} \;-\; \Gamma^{578\,10}\right). \tag{3.9}$$

The third projector, $\Pi_3 \;=\; \frac{1}{2}\left(\mathbb{1} \;-\; \Gamma^{569\,10}\right)$, is redundant given the other two. The choices of signs in these projectors are set by the choices of signs in the complex factors of (3.7). Note that the $(4, 11)$ index-pair is absent. These two projectors thus act in directions parallel to the brane moduli space. Imagine restricting the spinors to this slice of the metric: The brane moduli space has a Kähler 3-fold, and so it natural to reduce the spinors in terms of the $SU(3) \times U(1)$ holonomy, and isolate the supersymmetry as being the $SU(3)$ singlet. Imposing the requirement that $\Pi_1 \epsilon = \Pi_2 \epsilon = 0$ implements this. One can also see that this is equivalent to imposing the $SO(6)$ helicity conditions:

$$\Gamma^{69}\,\epsilon \;=\; \Gamma^{78}\,\epsilon \;=\; -\Gamma^{5\,10}\,\epsilon \;=\; \pm i\,\epsilon\,. \tag{3.10}$$

The form of these projectors on the moduli space is thus required by the underlying Kähler structure. The leap that we make in the general Ansatz is to assume, as we did in [6] that the projectors are unmodified as one moves off the space of moduli, and are given globally through the almost complex structure by (3.9).

The non-trivial task is to find the deformation of the standard $M2$-brane projector. We take this to be:

$$\begin{aligned}\Pi_0 \;&= \frac{1}{2}\Big(\mathbb{1} \;+\; \frac{2\,\kappa}{1+\kappa^2}\,\Gamma^{123} \;+\; \frac{1-\kappa^2}{1+\kappa^2}\,\big(\sin(\phi+3\psi)\Gamma^{4567\,11}+\cos(\phi+3\psi)\Gamma^{4568\,11}\big)\Big)\\ &= \frac{1}{2}\Big(\mathbb{1} \;+\; \frac{2\,\kappa}{1+\kappa^2}\,\Gamma^{123} \;-\; \frac{1-\kappa^2}{1+\kappa^2}\,\Gamma^{123}\,\big(\sin(\phi+3\psi)\Gamma^{89\,10}+\cos(\phi+3\psi)\Gamma^{79\,10}\big)\Big)\,.\end{aligned} \tag{3.11}$$

where $\kappa=\kappa(u,v)$ is an arbitrary function[1], and the second identity follows from $\Gamma^{1\ldots 11}=\mathbb{1}$,

While this projector seems rather complicated, it is in fact relatively simple to understand. One can argue simplify from the mathematical structure, but there is a very useful piece of physical input: The interpretation of (3.11) [16,17] should be associated with dielectric polarization of the $M2$ branes into $M5$ branes. This means that the deformation of the projector should involve the $M5$-brane component Γ^{123ABC} for some choice of A,B,C. Next, observe that Π_0 must commute with Π_1 and Π_2, and this is achieved in (3.11) by having A,B,C be exactly one out of each of the "complex pairs": $(6,9)$, $(7,8)$, $(5,10)$. There are $2^3=8$ such choices, however, the projection conditions, $\Pi_1\epsilon=\Pi_2\epsilon=0$, imply that only two of those choices are really independent, for example $\Pi_1\epsilon=0$ means that $\Gamma^{67}\epsilon=\Gamma^{89}\epsilon$ *etc.*. The two independent choices are those matrices appearing in (3.11).

Finally, to understand the angular dependence in (3.11) one needs to use the fact that the projectors must commute with the global symmetry generators.

3.4. Isometries and Lie derivatives

The metric has an $SU(3)\times U(1)^2$ isometry acting on the internal space. The $SU(3)$ acts transitively on the $\mathbb{CP}^2$ directions, while the $U(1)$'s are translations in ϕ and ψ. The tensor gauge field is obviously invariant under under a combination of these $U(1)$'s but it is, in fact, invariant under the complete $SU(3)\times U(1)\times U(1)$ group. To see the $U(1)$ invariances more explicitly, use the following expressions for the σ_j:

$$\begin{aligned}\sigma_1 \;&\equiv\; \cos\varphi_3\,d\varphi_1 \;+\; \sin\varphi_3\,\sin\varphi_1\,d\varphi_2\,,\\ \sigma_2 \;&\equiv\; \sin\varphi_3\,d\varphi_1 \;-\; \cos\varphi_3\,\sin\varphi_1\,d\varphi_2\,,\\ \sigma_3 \;&\equiv\; \cos\varphi_1\,d\varphi_2 \;+\; d\varphi_3\,,\end{aligned} \tag{3.12}$$

[1] We have chosen to use rational parametrization of the circular functions $\cos\chi$ and $\sin\chi$ in (1.5). This is largely because *Mathematica*^{TM} is more efficient with rational expressions.

from which one finds:

$$e^7 - i\,e^8 \;\sim\; e^{-i\varphi_3}\,(d\varphi_1 \;+\; i\,\sin\varphi_1\,d\varphi_2)\,. \tag{3.13}$$

Thus a translation in ψ or ϕ can be compensated in (3.7) by a translation in φ_3, or a rotation in the $(7,8)$-plane of the frames. This means that the $U(1)$ invariances of the background are generated by the Killing vectors, $L^\mu_{(1)}$ and $L^\mu_{(2)}$, where:

$$L^\mu_{(1)}\,\partial_\mu \;\equiv\; \frac{\partial}{\partial\varphi_3} \;+\; \frac{1}{3}\,\frac{\partial}{\partial\psi}\,, \qquad L^\mu_{(2)}\,\partial_\mu \;\equiv\; \frac{\partial}{\partial\phi} \;-\; \frac{1}{3}\,\frac{\partial}{\partial\psi}\,. \tag{3.14}$$

Given a Killing isometry of the metric, one can define the Lie derivative of a spinor field:

$$\begin{aligned} \mathcal{L}_K\,\epsilon \;&\equiv K^\mu\,\nabla_\mu\,\epsilon \;+\; \tfrac{1}{4}\,(\nabla_{[\mu}\,K_{\nu]})\,\Gamma^{\mu\nu}\,\epsilon \\ &= K^\mu\,\partial_\mu\,\epsilon \;+\; \tfrac{1}{4}\,(K^\rho\,\omega_{\rho\mu\nu} \;+\; \nabla_{[\mu}\,K_{\nu]})\,\Gamma^{\mu\nu}\,\epsilon\,. \end{aligned} \tag{3.15}$$

The factors are fixed by the requirement that this reduce to the usual Lie derivative on the vector $\bar\epsilon\Gamma^\mu\epsilon$.

A geometric symmetry of the solution must either act trivially on the Killing spinors, or it must be an $\mathcal{R}$-symmetry. For our flow, there is only a single $U(1)$ $\mathcal{R}$-symmetry, and this can determined which through perturbative analysis. We find that $L_{(2)}$ generates the non-trivial $\mathcal{R}$-symmetry while the Killiing spinor must be a singlet under the action of $L_{(1)}$. It is also important to note that the Killing spinor must be a singlet under the $SU(3)$ isometry. This means that

$$\mathcal{L}_K\,\epsilon \;\equiv\; 0\,. \tag{3.16}$$

when K is either $L_{(1)}$ or any $SU(3)$ generator.

It is trivial to compute the terms that make up the Lie derivative from the metric (3.4), and one finds that almost all the terms cancel between the connection and the curl of K_μ in (3.15). Indeed, one finds that:

$$\frac{\partial}{\partial y^\mu}\,\epsilon \;=\; 0\,, \qquad y^\mu \in \{\lambda, \varphi_1, \varphi_2, \varphi_3\}\,.$$

That is, the Killing spinor must be independent of the $\mathbb{CP}^2$ directions. From the $L_{(1)}$ action one obtains:

$$\frac{\partial}{\partial\psi}\,\epsilon \;=\; -\,\frac{3}{2}\,\Gamma^{78}\,\epsilon\,. \tag{3.17}$$

Moreover. if $\epsilon^{(a)}$, $a = 1, 2$ are the two Killing spinors, the $L_{(2)}$ action yields:

$$\begin{aligned} \Big(\frac{\partial}{\partial\phi} \ - \ \frac{1}{3}\,\frac{\partial}{\partial\psi}\Big)\,\epsilon^{(1)} \ &= \ -\,\epsilon^{(2)} \\ \Big(\frac{\partial}{\partial\phi} \ - \ \frac{1}{3}\,\frac{\partial}{\partial\psi}\Big)\,\epsilon^{(2)} \ &= \ +\,\epsilon^{(1)}\,. \end{aligned} \tag{3.18}$$

Having properly identified the symmetry actions on the spinors, one can then fix the complete angular dependence of the projection operator, Π_0, by requiring that it commute with these Lie derivative operators.

4. The new solutions

One now simply inserts the Ansatz into the gravitino variation, (1.1) and tries to solve all the equations. The system is hugely overdetermined, but at first sight is a little overwhelming. There are, however, some very useful simplifications.

First, there are combinations of the gravitino variations in which the tensor gauge fields cancel, leaving only metric terms:

$$\Gamma^1 \delta\psi_1 \ + \ \Gamma^7 \delta\psi_7 \ + \ \Gamma^8 \delta\psi_8 \ = \ 0\,, \qquad \Gamma^1 \delta\psi_1 \ + \ \Gamma^6 \delta\psi_6 \ + \ \Gamma^9 \delta\psi_9 \ = \ 0\,. \tag{4.1}$$

From these one immediately learns that

$$\begin{aligned} \partial_u\,(A_0 + 2\,A_3) \ &= 0\,, \qquad \partial_u\,(A_4 + 2\,A_2 - 2\,A_3) \ = \ 0\,, \\ v\,\partial_v\,(A_0 + 2\,A_3) \ &= 2\,(1 \ - \ e^{2\,A_2 - 2\,A_3 + A_4})\,. \end{aligned} \tag{4.2}$$

From this we see that $(A_4 + 2\,A_2 - 2\,A_3)$ is a function of v alone, however, because $e^5 = e^{A_2}\,dv$, there is the freedom to re-define A_2 up to an abritrary function of v, and hence arrange that $(A_4 + 2\,A_2 - 2\,A_3) = 0$. It then follows that $(A_0 + 2\,A_3) = const.$. We can scale the x_j and thereby set this constant to zero, but we will preserve it as an explicit scale. Thus these simple combinations of variations lead to:

$$A_0 \ = \ -2\,A_3 \ + \ 2\log(2\,L)\,, \qquad A_4 \ = \ -2\,(A_2 - A_3)\,, \tag{4.3}$$

where L is a constant scale factor.

After this the solution proceeds a little more slowly. The most direct procedure is to observe that there are only eight independent functions of u and v in the frame components,

F_{abcd}, of the field strength. One writes the variations in terms of these eight functions, and then carefully cross eliminates. This leads to the equations:

$$p_0 \;=\; -\frac{\kappa}{1+\kappa^2}\,, \qquad \frac{1-\kappa^2}{1+\kappa^2} \;=\; c\,u\,e^{4\,A_3-A_1}\,, \tag{4.4}$$

where c is a constant of integration. Using these facts, and continuing the cross-elimination yields:

$$p_2 \;=\; -\frac{1-\kappa}{1+\kappa}\,, \tag{4.5}$$

$$v\,\partial_v\,(A_1-A_2) \;=\; 3\,(e^{2\,(A_2-A_3)} \;-\; 1)\,. \tag{4.6}$$

At this point, the equations become rather complicated, but it is possible to disentangle them to obtain the following:

$$p_1 \;=\; -\,\frac{2\,v}{u}\,\frac{(1-\kappa^2)}{(1+\kappa^2)}\;e^{-A_1+A_2+A_5}\,, \tag{4.7}$$

$$u\,\partial_u\,(A_1-A_2) \;=\; -3\,e^{2\,A_2-2\,A_3+A_5}\,, \tag{4.8}$$

$$\partial_u\,(A_1-A_2) \;=\; \frac{3\,v}{4\,u}\,\partial_v\left[\frac{(1+\kappa)^2}{(1+\kappa^2)}\;e^{2\,(A_1-A_3)}\right], \tag{4.9}$$

$$u\,\partial_u\left[\frac{(1+\kappa)^2}{(1+\kappa^2)}\;e^{2\,(A_1-A_3)}\right] \;=\; -2\,\frac{(1-\kappa)^2}{(1+\kappa^2)}\;e^{2\,(A_1-A_3)}\,. \tag{4.10}$$

This system of equations suffices to solve everything. Define:

$$g \equiv 2\,(A_1-A_2)\,, \qquad h \equiv \frac{(1+\kappa)^2}{(1+\kappa^2)}\;e^{2\,(A_1-A_3)}\,, \tag{4.11}$$

then the foregoing imply:

$$\partial_u\,g \;=\; \frac{3}{2}\,\frac{u}{v}\,\partial_v\,h\,, \tag{4.12}$$

$$\begin{aligned} u\,\partial_u\,h \;-\; 2\,h \;&=\; -\,4\,e^{2\,(A_1-A_3)} \\ &=\; -\,\frac{2}{3}\,\frac{1}{v^5}\,\partial_v\big(v^6\,e^g\big)\,. \end{aligned} \tag{4.13}$$

Eliminating h from these two equations yields the "master equation" for g:

$$u^3\,\partial_u\,\big(u^{-3}\,\partial_u\,g\big) \;+\; v^{-1}\,\partial_v\,\big(v^{-5}\,\partial_v\big(v^6\,e^g\big)\big) \;=\; 0\,. \tag{4.14}$$

This may also be written:

$$\partial_u^2\,g \;-\; 3\,u^{-1}\partial_u^2\,g \;+\; \big(\partial_v^2\,g \;+\; 7\,v^{-1}\,\partial_v\,g \;+\; (\partial_v\,g)^2\big)\,e^g \;=\; 0\,. \tag{4.15}$$

Suppose that one has a solution, g, to this equation; that is, one knows $(A_1 - A_2)$. One can then obtain $(A_2 - A_3)$ and A_5 from (4.6) and (4.8) by differentiating g. From (4.9) and (4.13) one can then obtain $\partial_v(u^{-2}h)$ and $\partial_u(u^{-2}h)$ respectively, and by quadrature one thus obtains κ. From this and (4.4) one obtains $4A_3 - A_1$, and hence A_1, A_2 and A_3 independently. Finally, the p_j and A_0 and A_4 are obtained from (4.4), (4.7), (4.5) and (4.3). In this procedure almost every function is obtained directly from g and its derivatives. The only exception is that h is obtained by quadrature. Thus having solved (4.14), all other parts of the solution are obtained by elementary operations: At no point do we have to solve any further equations. It is in this sense that (4.14) is the master equation of the new family of solutions.

So far we have solved the Killing spinor equations. As was pointed in [2,6] this is not enough to guarantee that the solution we found also satisfies the equation of motion. The reason for this is that the commutator of two supersymmetries does not necessarily generate *all* the equations of motion. An analysis of precisely which subset of the equations of motion are generated may be found in [2], and using this one can significantly reduce the number of equations that need to be verified.

On the other hand, the hard work has largely been done in solving the supersymmetry variations, and the difficulty of verifying all the equations of motion is not overly burdensome, and so we checked them all for completeness. With (3.4) there are eight non-trivial Einstein equations. There are six independent diagonal terms[2] (with $\mu = \nu$), and two non-diagonal ones with $(\mu = 4, \nu = 5)$ and $(\mu = 10, \nu = 11)$. All of these Einstein equations can be expressed in terms of the relations (4.3)–(4.14) found by solving the Killing spinor equations or their u and v derivatives, and so our solution satisfies all the Einstein equations.

The Maxwell equations can be verified in a similar fashion. There are many more terms to be checked but again after lengthy but straight forward algebraic manipulations one finds, once again, that the Maxwell equations can be expressed in terms of the relations (4.3)–(4.14) and their u and v derivatives. Thus the PDE (4.14) is both sufficient and necessary for solving all the equation of motion.

[2] Here we are using the Poincaré symmetry on the brane and the $SU(3)$ symmetry on the $\mathbb{CP}^2$.

5. G-structures and simplifications

The foregoing analysis was done by working directly with the supersymmetry transformations. In practice it is non-trivial to extract the simple equations from (1.1). This entire process can be sometimes be simplified by use of some of the G-structure identities.

First, the vectors

$$K^{\mu}_{(ij)} \;\equiv\; \bar{\epsilon}^{(i)}\,\Gamma^{\mu}\epsilon^{(j)}\,, \tag{5.1}$$

are always Killing vectors, and usually one finds that if these are non-zero then they are translations to the brane. There are examples where one generates other Killing vectors [6,7,17], but here we find that the only Killing vectors obtained from (5.1) are indeed those along the brane.

The other very useful identities involve the 2-forms:

$$\Omega^{(ij)}_{\mu\nu} \;=\; \epsilon^{(i)}\,\Gamma_{\mu\nu}\epsilon^{(j)}\,, \tag{5.2}$$

which satisfy simple differential identities as a consequence of (1.1). If one skew symmetrizes such an identity, it reduces to:

$$\partial_{[\rho}\Omega^{(ij)}_{\mu\nu]} \;=\; F_{\rho\mu\nu\sigma}\,K^{\sigma}_{(ij)}\,. \tag{5.3}$$

An immediate application of this to observe that if the Killing vector components, $K^{\mu}_{(ij)}$, are in fact constant, and the field $F_{\rho\mu\nu\sigma}$ is independent of the coordinates dual to these Killing vectors, then there is a gauge in which:

$$\Omega^{(ij)}_{\mu\nu} \;=\; \tfrac{1}{3}\,A^{(3)}_{\mu\nu\rho}\,K^{\rho}_{(ij)}\,. \tag{5.4}$$

This determines some of the components of $A^{(3)}$, and in particular, leads immediately from the projector (3.11) to the expression for p_0 in (4.4).

Suppose now that $\epsilon^{(i)} = \epsilon^{(j)} = \epsilon$, and define

$$K^{\mu} \;\equiv\; \bar{\epsilon}\,\Gamma^{\mu}\epsilon\,, \qquad \Omega_{\mu\nu} \;\equiv\; \bar{\epsilon}\,\Gamma_{\mu\nu}\epsilon\,. \tag{5.5}$$

One can then derive [2] the algebraic identities:

$$\Omega_{\mu}{}^{\rho}\,\Omega_{\rho}{}^{\nu} \;=\; (K^2)\,\delta^{\nu}_{\mu} \;-\; K_{\mu}\,K^{\nu}\,, \qquad K^{\mu}\Omega_{\mu\nu} \;=\; 0\,. \tag{5.6}$$

Indeed, by suitable choice of ϵ, one can arrange that K^{μ} is simply $K^{\mu} = \delta^{\mu}_0$, and hence:

$$\Omega_{b}{}^{c}\,\Omega_{c}{}^{a} \;=\; e^{2A_0}\,\delta^{a}_{b}\,, \qquad a,b,c = 2,\ldots,11\,. \tag{5.7}$$

Therefore, the ten-dimensional, spatial metric is has an almost complex structure, and the obvious question is whether it is a complex, or even Kähler structure. More precisely, if one conformally rescales the metric by e^{-2A_0} then $\Omega_{\mu\nu}$ provides an almost complex structure satisfying (5.3), which means that it is closed except for components parallel to the brane.

If one uses the metric Ansatz (3.4) and imposes the projection conditions using (3.9) and (3.11), then the 2-form, Ω, becomes:

$$\begin{aligned}\Omega \;&= e^{A_0}\Big[\frac{2\,\kappa}{1+\kappa^2}\,(e^2\wedge e^3 \;+\; e^4\wedge e^{11}) \;+\; e^6\wedge e^9 \;+\; e^7\wedge e^8 \;-\; e^5\wedge e^{10} \\ &+\; \frac{1-\kappa^2}{1+\kappa^2}\,((-\cos(2\,\phi)\,e^2+\sin(2\,\phi)\,e^3)\wedge e^4+(\cos(2\,\phi)\,e^3+\sin(2\,\phi)\,e^2)\wedge e^{11})\Big]\,.\end{aligned} \tag{5.8}$$

Observe that this contains and extends the Kähler structure of the moduli space (3.8) to the entire spatial section of the metric. The fact that the G-structure must contain such a form, Ω, gave us the canonical way to extend the special holonomy projectors away from the moduli space. One should also note how the non-trivial deformation of the projector Π_0 manifests itself here in terms of a nontrivial mixing, or fibering of the spatial directions of the brane over the (u,ϕ) directions. Moreover, this mixing is precisely what spoils the global Kähler structure: One cannot project Ω perpendicular to the branes and preserve the projection of (5.7).

One can also derive many useful identities from (5.8) and (5.3). As we have already remarked, the equations parallel to the brane lead to p_0 as given in (4.4). The components parallel to the $\mathbb{CP}^2$ factor lead a system like (4.2), and which reduces to (4.3) after fixing the remaining coordinate invariance. The components in the directions (u,ϕ,x^j) and (v,ϕ,x^j) then lead to the second equation in (4.4). Finally, the closure in the (u,v,ϕ) direction leads to the differential equation:

$$u\,\partial_v\left[\frac{2\,\kappa}{(1+\kappa^2)}\;e^{2\,(A_1-A_3)}\right]\;+\;v\,\partial_u\left(e^{2\,A_2-2\,A_3+A_5}\right)\;=\;0\,, \tag{5.9}$$

which is a combination of the equations found in section 4.

The construction of the forms, $\Omega^{(ij)}$, was also very useful in finding the coordinate transformations (3.5) for the solution of [6]. In particular, by constructing (5.8) and projecting parallel to the brane one can read off the complex pair of differentials $(du,d\phi)$.

6. Final Comments

We have shown that the "algebraic Killing spinor" technique can be adapted to problems with four supersymmetries. In this paper, and in [6] we were able to deduce the projectors that define the Killing spinors through a combination of two ideas. First, one looks at the supersymmetry conditions on the brane-probe moduli space, where the metric is either Kähler, or hyper-Kähler. On this space the supersymmetries are obtained via standard holonomy techniques, and the resulting projectors can be extended to the complete (spatial) metric via the almost complex structure. The second idea is that while the original holographic theory is based on some standard brane configuration, the non-trivial deformation involves a dielectric deformation of this configuration into some five-branes. The canonical supersymmetry projector must reflect this deformation, and its form can be fixed from this and the fact that it must commute with the symmetry generators and with the other projectors.

We have now obtained quite a number of families of solutions that are motivated by physically important RG flows and yet involve this dielectric deformation of the underlying branes [6,7,8,16,17]. These solutions all involve a moduli space of branes, typically a Coulomb branch of some non-trivial flow, and they describe the spreading of the branes in a symmetric manner. These solutions thus generalize the usual harmonic brane Ansatz. Indeed, one is led to solutions that are characterized by a single function, but that function is determined by a non-linear PDE. Here the "master equation" is (4.14), while in [6] it was:

$$\frac{1}{u^3}\frac{\partial}{\partial u}\Big(u^3\frac{\partial}{\partial u}\Big(\frac{1}{u^2}\,f(u,v)\Big)\Big) \;+\; \frac{1}{v}\frac{\partial}{\partial v}\Big(f(u,v)\,\frac{1}{v}\,\frac{\partial}{\partial v}\,\Big(\frac{v^2}{u^2}\,f(u,v)\Big)\Big) \;=\; 0\,. \tag{6.1}$$

At first sight this seems somewhat different, but one can reduce to an equation of the same form as (4.14) by setting $f(u,v) = e^{g(u,v)}$ and then canceling a factor of $e^{g(u,v)}$ from each side of the equation. While these equations are non-linear, they do have a relatively simple, linear perturbation expansion of the form (1.3) [6]. In particular, at leading order there is a "seed function" that satisfies a homogenous, linear PDE that presumably encodes the distribution of branes on the moduli-space.

More generally, our projector Ansatz (3.9) and (3.11) represents a rather simple modification of the corresponding Ansatz on a Calabi-Yau manifold, particularly given the appearance of a ten-dimensional almost complex structure in our solution.

The evolution of of the work presented here, and in [6,7,8,16,17], is perhaps somewhat reminiscent of the story of "warp factors." The latter originally emerged from the technicalities of understanding how lower-dimensional supergravity theories were embedded

within higher-dimensional theories, but in recent years, warp factors have entered into the mainstream of string compactification, and even into phenomenology. In this paper, and our earlier ones, we started by trying to understand how supersymmetry was broken in a class of solutions coming from lower-dimensional supergravity only to find that, in higher dimensions, these solutions can be generalized to whole families and that these families all necessarily involve dielectric deformations of the original branes. It would therefore not surprise us if the classes of solutions we are considering will be of rather broader significance in the study of supersymmetry breaking and supersymmetric backgrounds.

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